Beyond i.i.d. in the Resource Theory of Asymmetry: An Information-Spectrum Approach for Quantum Fisher Information

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[KY and Hiroyasu Tajima, arXiv:2204.08439]

Energetic coherence as a resource

Energetic coherence
= superposition between eigenstates of the Hamiltonian with different eigenvalues.

This resource is mandatory for

- Creating accurate clocks
- Accelerating quantum operations
- Measuring physical quantities that do not commute with the Hamiltonian

The resource theory of asymmetry (RTA) is a branch of resource theories that investigates the sequence of the symmetry of the dynamics and conservation laws.

Resource manipulation in RTA

In any resource theory, resource manipulation (= dilution and distillation) is an essential.

Several important results are known in RTA. For example,

- Exact convertibility among pure states [G. Gour and R. W. Spekkens (2008), I. Marvian PhD thesis (2012)]
- Asymptotic conversion theory for i.i.d. states $\phi^{\otimes m} \to \psi^{\otimes n}$ [I. Marvian (2022)]

However, resource conversion in the non-i.i.d. regime has not been established in RTA.

Non-i.i.d. theories in entanglement theory

Non-i.i.d. theories are established e.g., in the resource theory of entanglement.

This is established with the information-spectrum method.

The information-spectrum method is a powerful tool to analyze the non-i.i.d. regime for problems related to entropy.

Ent. cost: $E_{\mathbf{cost}}$ =(the minimal rate of Bell states required to crate a sequence of states)

Dist. ent.: $E_{\mathbf{dist}}$ = (the maximal rate of Bell states extractable from a sequence of states)

For **any** sequence of pure states $\hat{\psi}$, they are given by the spectral sup- and inf-entropy rates $\overline{S}$, $\underline{S}$

$$E_{\mathrm{cost}}(\hat{\psi}) = \overline{S}(\hat{\rho})$$

[G. Bowen and N. Datta (2008)]

$$E_{\mathrm{dist}}(\hat{\psi}) = \underline{S}(\hat{\rho})$$

[M. Hayashi (2003)]

Non-i.i.d. theory for RTA?

So far, it has not been possible to apply the information-spectrum method to RTA.

This is because a standard measure of energetic coherence in RTA is
the quantum Fisher information (QFI), which is quite different from entropy.

We here propose an information-spectrum approach for QFI to establish non-i.i.d. theory in RTA.

Main achievements

[KY and Hiroyasu Tajima, arXiv:2204.08439]

Main achievements are three:

1. We introduce new quantities, termed **the spectral sup- and inf-QFI rates**

   [They are the counterparts of the spectral entropy rates.]

2. To construct the spectral sup- and inf-QFI rates through the smoothing method, we define **the max- and min-QFI**

   [They are the counterparts of max- and min-entropies]

3. To show the properties of the max- and min-QFI, we introduce the notion of **asymmetric majorization** for probability distributions. We show that the exact convertibility between pure states in RTA is expressed by an asymmetric majorization relation.

   [This is the counterpart of Nielsen's theorem]

Outline of this talk

- Introduction
- Resource theory of asymmetry
- The spectral QFI rates
- Main theorem:
    - the coherence cost, the distillable coherence and the spectral QFI rates
- Intuitive explanation of the main theorem

The resource theory of asymmetry (RTA)

In RTA, we consider a quantum system with a Hamiltonian H.

Free operation = covariant operation

$\mathcal{E}$ is called covariant iff it commutes with the time-translation, i.e.,

$$\mathcal{E}\left(e^{-iHt}\rho e^{IHt}\right) = e^{-iHt}\mathcal{E}(\rho)e^{iHt}$$

Free state = symmetric state = state w/o energetic coherence*

ρ is called symmetric iff $[\rho, H] = 0$

* Energetic coherence: superposition between energy eigenstates w/ different energies

Resource state = asymmetric state = state w/ energetic coherence

ρ is called asymmetric iff $[\rho, H] \neq 0$

QFI as a standard measure

A resource measure R satisfies

1. $R(\rho) \geq R(\mathcal{E}(\rho))$ for any free operation $\mathcal{E}$
2. $R(\rho) = 0$ for any free state ρ

A crucial resource measure in RTA is the symmetric logarithmic derivative quantum Fisher information w.r.t. $\rho_t := \{e^{-\mathrm{i}Ht}\rho e^{\mathrm{i}Ht}\}_t$, given by

$$\mathcal{F}(\rho) = 2\sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |\langle i|H|j\rangle|^2$$

Eigenvalue decomposition:

$$\rho = \sum_i \lambda_i |i\rangle\langle i|$$

Conventionally, this quantity is called the Quantum Fisher information (QFI).

Hamiltonian for harmonic oscillators

From now on, we assume that the Hamiltonian is given by the harmonic oscillator Hamiltonian for simplicity.

$$H = \sum_{n=0}^{\infty} n\,|n\rangle\langle n|$$

Conversion theory for harmonic oscillators in pure states can be generalized to any systems in periodic pure states with an arbitrary Hamiltonian.

[I. Marvian, arXiv:2112.04694]

[KY and Hiroyasu Tajima, arXiv:2204.08439]

Asymptotic convertibility

We adopt the trace distance $D(\rho,\sigma) := \frac{1}{2}\|\rho-\sigma\|_1$ as a quantifier of error.

We say that a sequence $\hat{\rho} = \{\rho_m\}_m$ is convertible to another sequence $\hat{\sigma} = \{\sigma_m\}_m$ by cov. ops. iff

$$\exists \text{ covariant operations } \{\mathcal{E}_m\}_m \text{ s.t. } \lim_{m\to\infty} D(\mathcal{E}_m(\rho_m),\sigma_m) = 0.$$

In this case, we denote $\hat{\rho} \overset{\mathrm{cov}}{\succ} \hat{\sigma}$.

We introduce two key quantities: the coherence cost and the distillable coherence

$$C_{\mathrm{cost}}(\hat{\rho}) := \inf\left\{R \,\middle|\, \widehat{\phi_{\mathrm{coh}}}(R) \overset{\mathrm{cov}}{\succ} \hat{\rho}\right\} \qquad C_{\mathrm{dist}}(\hat{\rho}) := \sup\left\{R \,\middle|\, \hat{\rho} \overset{\mathrm{cov}}{\succ} \widehat{\phi_{\mathrm{coh}}}(R)\right\}$$

Coherence bit: $|\phi_{\mathrm{coh}}\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\phi_{\mathrm{coh}} := |\phi_{\mathrm{coh}}\rangle\langle\phi_{\mathrm{coh}}|$, $\widehat{\phi_{\mathrm{coh}}}(R) := \left\{\phi_{\mathrm{coh}}^{\otimes\lceil Rm\rceil}\right\}_m$

In the i.i.d. regime, $C_{\mathrm{cost}}(\hat{\psi}) = C_{\mathrm{dist}}(\hat{\psi}) = \mathcal{F}(\psi)$ holds for $\hat{\psi} = \{\psi^{\otimes m}\}_m$ with a pure state ψ with period 2π.

[I. Marvian (2022)]

Notations

Energy distribution:

For a given pure state ψ, we denote $p_\psi(n) \coloneqq |\langle n|\psi\rangle|^2$.

Product (convolution) $*$:

For two sequences of numbers $a = \{a(n)\}_n$ and $b = \{b(n)\}_n$, we define their product $a * b$ by

$$a * b(n) \coloneqq \sum_{k\in\mathbb{Z}} a(k)b(n-k)$$

Inverse sequence $\tilde{q}$:

For a given sequence q, we say another sequence $\tilde{q}$ is its inverse when it satisfies $\tilde{q} * q(n) = \delta_{0,n}$.

If there exists $n_* = \min\{n|q(n) > 0\}$, then the unique $\tilde{q}$ can be explicitly constructed by a recursive formula.

The inverse sequence for energy distributions is essential to construct the spectral QFI rates.

Generalized Poisson distribution

Generalized Poisson distribution:

For $\lambda \in \mathbb{R}$, we define $\mathrm{P}_\lambda = \{\mathrm{P}_\lambda(n)\}$ by $\mathrm{P}_\lambda(n) := \begin{cases} e^{-\lambda}\dfrac{\lambda^n}{n!} & (n \geq 0) \\ 0 & (n < 0) \end{cases}$

For $\lambda \geq 0$, it is an ordinary Poisson distribution.
For $\lambda < 0$, it is **not** a probability distribution. Nevertheless, it plays an important role since $\widetilde{\mathrm{P}_\lambda} = \mathrm{P}_{-\lambda}$

The max- and min-QFI

We introduce two key quantities for a pure state ψ:

$$\mathcal{F}_{\max}(\psi) \coloneqq \inf\left\{4\lambda \middle| \mathrm{P}_\lambda * \widetilde{p_\psi} \geq 0\right\} \qquad \mathcal{F}_{\min}(\psi) \coloneqq \sup\left\{4\lambda \middle| p_\psi * \mathrm{P}_{-\lambda} \geq 0\right\}$$

The max- and min-QFI are the amounts of energetic coherence in ψ that can be transformed from and to a pure state whose energy distribution is given by the Poisson distribution.

The max-QFI is also defined for a general state ρ by $\mathcal{F}_{\max}(\rho) \coloneqq \inf_{\Phi_\rho, H_A} \mathcal{F}_{\max}(\Phi_\rho)$

(Φ_ρ: purification of ρ, H_A: the Hamiltonian of ancilla w/ integer eigenvalues)

The max- and min-QFIs have similar properties to the max- and min-entropies. For example,

$$\mathcal{F}_{\max}(\psi) \geq \mathcal{F}(\psi) \geq \mathcal{F}_{\min}(\psi)$$

[KY and Hiroyasu Tajima, arXiv:2204.08439]

The spectral QFI rates

We define the spectral sup-and inf-QFI rates by

$$\overline{\mathcal{F}}(\hat{\psi}) \coloneqq \lim_{\epsilon\to 0} \limsup_{m\to\infty} \frac{1}{m} \mathcal{F}^{\epsilon}_{\max}(\psi_m) \qquad \underline{\mathcal{F}}(\hat{\psi}) \coloneqq \lim_{\epsilon\to 0} \liminf_{m\to\infty} \frac{1}{m} \mathcal{F}^{\epsilon}_{\min}(\psi_m)$$

where the smooth max- and min-QFIs are defined by

$$\mathcal{F}^{\epsilon}_{\max}(\psi) \coloneqq \inf_{\rho\in B^{\epsilon}(\psi)} \mathcal{F}_{\max}(\rho) \qquad \mathcal{F}^{\epsilon}_{\min}(\psi) \coloneqq \sup_{\rho\in B^{\epsilon}_{\mathrm{pure}}(\psi)} \mathcal{F}_{\min}(\rho)$$

$$B^{\epsilon}(\rho) \coloneqq \{\text{states } \rho | D(\rho,\sigma) \le \epsilon\} \qquad B^{\epsilon}_{\mathrm{pure}}(\rho) \coloneqq \{\text{pure states } \phi | D(\rho,\phi) \le \epsilon\}$$

[KY and Hiroyasu Tajima, arXiv:2204.08439]

Cf. The spectral entropy rates w/ smoothing method:

$$\overline{S}(\hat{\rho}) \coloneqq \lim_{\epsilon\to 0} \limsup_{m\to\infty} \frac{1}{m} S^{\epsilon}_{\max}(\psi_m) \qquad \underline{S}(\hat{\rho}) \coloneqq \lim_{\epsilon\to 0} \liminf_{m\to\infty} \frac{1}{m} S^{\epsilon}_{\min}(\psi_m)$$

$$S^{\epsilon}_{\max}(\psi) \coloneqq \inf_{\rho\in B^{\epsilon}(\psi)} S_{\max}(\rho) \qquad S^{\epsilon}_{\min}(\psi) \coloneqq \sup_{\rho\in B^{\epsilon}(\psi)} S_{\min}(\rho)$$

[N. Datta and R. Renner (2009)]
[R. Renner, PhD thesis (2005)]

Main theorem

Main result [KY and Hiroyasu Tajima, arXiv:2204.08439]

For **any** sequence of pure states $\hat{\psi} = \{\psi_m\}_m$,

$$C_{\text{cost}}(\hat{\psi}) = \overline{\mathcal{F}}(\hat{\psi}) \qquad C_{\text{dist}}(\hat{\psi}) = \underline{\mathcal{F}}(\hat{\psi})$$

The spectral QFI rates are defined by $\overline{\mathcal{F}}(\hat{\psi}) := \lim_{\epsilon\to 0} \limsup_{m\to\infty} \frac{1}{m} \mathcal{F}_{\max}^{\epsilon}(\psi_m)$, $\underline{\mathcal{F}}(\hat{\psi}) := \lim_{\epsilon\to 0} \liminf_{m\to\infty} \frac{1}{m} \mathcal{F}_{\min}^{\epsilon}(\psi_m)$

In entanglement theory

For **any** sequence of pure states $\hat{\psi} = \{\psi_m\}_m$,

$$E_{\text{cost}}(\hat{\psi}) = \overline{S}(\hat{\rho}) \qquad E_{\text{dist}}(\hat{\psi}) = \underline{S}(\hat{\rho}) \quad (\hat{\rho} = \{\rho_m\},\ \ \rho_m = \mathrm{Tr}_B(\psi_{AB,m}))$$

[M. Hayashi (2003), G. Bowen and N. Datta (2008)]

The spectral entropy rates are given by $\overline{S}(\hat{\rho}) := \lim_{\epsilon\to 0} \limsup_{m\to\infty} \frac{1}{m} S_{\max}^{\epsilon}(\rho_m)$, $\underline{S}(\hat{\rho}) := \lim_{\epsilon\to 0} \liminf_{m\to\infty} \frac{1}{m} S_{\min}^{\epsilon}(\rho_m)$

[N. Datta and R. Renner (2009)]

Rewriting $C_{\mathrm{cost}}(\hat{\psi})$ w/ Poisson distr.

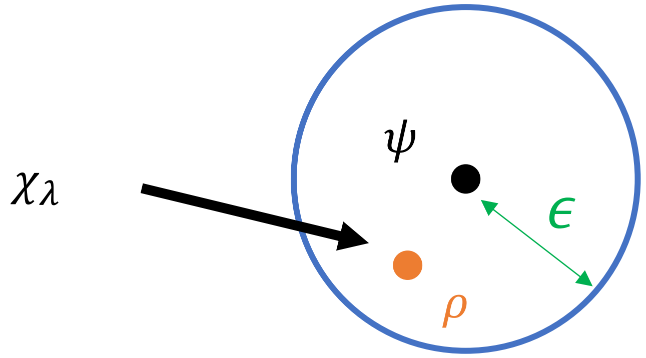
- The energy distr. of $\phi_{\mathrm{coh}}^{\otimes[Rm]}$ converges to a Poisson distr. $\mathrm{P}_{Rm/4}$ up to a shift as $m \to \infty$
- The energy can be shifted by covariant operations

$\longrightarrow$ $\left\{\phi_{\mathrm{coh}}^{\otimes[Rm]}\right\}_m$ and $\{\chi_{Rm/4}\}_m$ are asymptotically interconvertible, where $|\chi_\lambda\rangle \coloneqq \sum_n \sqrt{\mathrm{P}_\lambda(n)}|n\rangle$

$$C_{\mathrm{cost}}(\hat{\psi}) \coloneqq \inf\left\{R \middle| \widehat{\phi_{\mathrm{coh}}}(R) \overset{\mathrm{cov}}{\succ} \hat{\psi}\right\} = \inf\left\{R \middle| \exists \mathrm{cov.\,ops.}\, \{\mathcal{E}_m\}_m \text{ s.t. } \lim_{m\to\infty} D\left(\mathcal{E}_m\left(\phi_{\mathrm{coh}}^{\otimes[Rm]}\right), \psi_m\right) = 0\right\}$$

$$= \inf\left\{4\lambda \middle| \exists \mathrm{cov.\,ops.}\, \{\mathcal{E}_m\}_m \text{ s.t. } \lim_{m\to\infty} D(\mathcal{E}_m(\chi_{m\lambda}), \psi_m) = 0\right\}$$

Target: $\inf\{4\lambda | \exists \mathrm{cov.\,op.}\, \mathcal{E} \text{ s.t. } \mathcal{E}(\chi_\lambda) = \rho\}$ for $\rho \in B^\epsilon(\psi)$



We will show $\mathcal{F}_{\max}(\rho) = \inf\{4\lambda | \exists \mathrm{cov.\,op.}\, \mathcal{E} \text{ s.t. } \mathcal{E}(\chi_\lambda) = \rho\}$

a-majorization

We here introduce a notion of asymmetric-majorization (a-majorization).

For given two probability distributions $p = \{p(n)\}_n$ and $q = \{q(n)\}_n$, we say that p a-majorizes q iff

$$p * \tilde{q}(n) \geq 0 \text{ for all } n \in \mathbb{Z}$$

In this case, we denote $p \succ_a q$.

A key result:

A pure state ψ is convertible to ϕ by a covariant operation w/o error iff $\quad p_\psi \succ_a p_\phi$

[KY and Hiroyasu Tajima, arXiv:2204.08439]

[Other forms of NS condition for the exact conversion can be found e.g., in G. Gour and R. W. Spekkens (2008)]

(cf.) Nielsen's theorem in entanglement theory:

A pure state ψ is convertible to ϕ by a LOCC w/o error iff $\quad \lambda_\psi \prec \lambda_\phi$

λ_ψ: the prob. distr. defined by the Schmidt coefficients of a bipartite pure state ψ_{AB}

max-QFI for pure states

The max-QFI for pure state ψ is defined by

$$\mathcal{F}_{\max}(\psi) := \inf\left\{4\lambda | \mathrm{P}_\lambda * \widetilde{p_\psi} \geq 0\right\} \qquad \text{(a-majorization)}$$
$$= \inf\{4\lambda | \exists \mathrm{cov.op.}\,\mathcal{E}\ \mathrm{s.t.}\ \mathcal{E}(\chi_\lambda) = \psi\}$$

This is exactly what we need for calculating the coherence cost.

Since the QFI for $|\chi_\lambda\rangle = \sum_n \sqrt{\mathrm{P}_\lambda(n)}|n\rangle$ is given by $\mathcal{F}(\chi_\lambda) = 4\lambda$, we can also rewrite

$$\mathcal{F}_{\max}(\psi) = \inf\{\mathcal{F}(\chi_\lambda) | \exists \mathrm{cov.op.}\,\mathcal{E}\ \mathrm{s.t.}\ \mathcal{E}(\chi_\lambda) = \psi\}$$

$\mathcal{F}_{\max}$ = The minimal amount of energetic coherence (i.e., QFI) in $|\chi_\lambda\rangle$ that is required to create $|\psi\rangle$.

$\mathcal{F}_{\min}$ = The maximum amount of energetic coherence in $|\chi_\lambda\rangle$ that can be extracted from $|\psi\rangle$.

max-QFI for mixed states

$$\mathcal{F}_{\max}(\rho) \coloneqq \inf_{\Phi_\rho, H_A} \mathcal{F}_{\max}(\Phi_\rho)$$

Let Φ_ρ denote a purification of a general state ρ.

If ∃ cov. op. $\mathcal{E}$ s.t. $\mathcal{E}(\chi_\lambda) = \Phi_\rho$, then ρ can be created from χ_λ since $\mathrm{tr}_R \circ \mathcal{E}(\chi_\lambda) = \rho$.

$$\longrightarrow \quad \inf_{\Phi_\rho, H_A} \mathcal{F}_{\max}(\Phi_\rho) \geq \inf\{4\lambda | \exists \text{cov. op.} \mathcal{E} \text{ s.t.} \mathcal{E}(\chi_\lambda) = \rho\}$$

If ∃ cov. op. $\mathcal{E}'$ s.t. $\mathcal{E}'(\chi_\lambda) = \rho$, then ∃ a purification Φ_ρ that can be created from χ_λ.

cov. Stinespring dilation: For any covariant operation $\mathcal{E}'$, ∃ an ancilla A with Hamiltonian H_A, its eigenstate $|\eta_A\rangle$ and an energy-preserving (covariant) unitary U_{SA} s.t. $\mathcal{E}'(\cdots) = Tr_A\left(U_{SA}(\cdots \otimes |\eta_A\rangle\langle\eta_A|)U_{SA}^\dagger\right)$

$$\longrightarrow \quad \inf_{\Phi_\rho, H_A} \mathcal{F}_{\max}\left(\Phi_\rho\right) \leq \inf\{4\lambda | \exists \text{cov. op.} \mathcal{E} \text{ s.t.} \mathcal{E}(\chi_\lambda) = \rho\}$$

Therefore, $\mathcal{F}_{\max}(\rho) = \inf\{4\lambda | \exists \text{cov. op.} \mathcal{E} \text{ s.t.} \mathcal{E}(\chi_\lambda) = \rho\}$

Since $C_{\mathrm{cost}}(\hat{\psi}) = \inf\left\{4\lambda | \exists \text{cov. ops.} \{\mathcal{E}_m\}_m \text{ s.t.} \lim_{m\to\infty} D(\mathcal{E}_m(\chi_{m\lambda}), \psi_m) = 0\right\}$, we get

$$C_{\mathrm{cost}}(\hat{\psi}) = \overline{\mathcal{F}}(\hat{\psi}) \coloneqq \lim_{\epsilon\to 0} \limsup_{m\to\infty} \frac{1}{m} \mathcal{F}_{\max}^{\epsilon}(\psi_m)$$, where $$\mathcal{F}_{\max}^{\epsilon}(\psi) \coloneqq \inf_{\rho \in B^\epsilon(\psi)} \mathcal{F}_{\max}(\rho)$$

See [KY and Hiroyasu Tajima, arXiv:2204.08439] for further details.

Summary

We established asymptotic conversion theory between pure states in the non-i.i.d. regime by constructing the spectral sup- and inf-QFI rates.

$$C_{\mathrm{cost}}(\hat{\psi}) = \overline{\mathcal{F}}(\hat{\psi}) \qquad \overline{\mathcal{F}}(\hat{\psi}) \coloneqq \lim_{\epsilon\to 0} \limsup_{m\to\infty} \frac{1}{m} \mathcal{F}_{\max}^{\epsilon}(\psi_m)$$

$$C_{\mathrm{dist}}(\hat{\psi}) = \underline{\mathcal{F}}(\hat{\psi}) \qquad \underline{\mathcal{F}}(\hat{\psi}) \coloneqq \lim_{\epsilon\to 0} \liminf_{m\to\infty} \frac{1}{m} \mathcal{F}_{\min}^{\epsilon}(\psi_m)$$

To construct the spectral sup- and inf-QFI rates through the smoothing method, we define the max- and min-QFI.

Asymmetric majorization relation gives a necessary and sufficient condition for exact convertibility among pure states, which is the counterpart in RTA to Nielsen's theorem.