General Quantum Resource Theories

Maximal Resources, Catalytic Replication, and Asymptotically Consistent Measures

Kohdai Kuroiwa [1], Hayata Yamasaki [2]
1: University of Waterloo, 2: The University of Tokyo

hayata.yamasaki@gmail.com @hayatayamasaki

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https://arxiv.org/abs/2002.02458 K. Kuroiwa and H. Yamasaki, Quantum 4, 355 (2020)
https://arxiv.org/abs/2103.05665 K. Kuroiwa and H. Yamasaki, Physical Review A Letter 104, L020401 (2021)

Quantum resource theories (QRTs)

A unified framework for analyzing inherent quantum properties

Quantum properties = Resources to overcome restrictions on quantum operations

Example: entanglement, magic states, non-Gaussianity,…

Central topics:

- **Manipulation**: How can we transform resources by free operations?
- **Quantification**: How can we quantify resources?

E. Chitambar, G. Gour, Rev. Mod. Phys. 91, 025001 (2019)

Challenge in Establishing General QRTs

What are universal properties of quantum resources shared among QRTs?

General QRTs: Investigating universal properties shared among many resources

Previous works on general QRTs with **limited applicability**

Mathematical assumptions have been imposed to make the analysis tractable

e.g. uniqueness of a maximal resourceful state, convexity, finite-dimensionality

E.g., Z.W. Liu et al. (2019), R. Takagi et al. (2019), B. Regula et al. (2020),…

Challenge: **Physically well-motivated resources** may not satisfy assumptions to make analysis tractable

- **Non-uniqueness of max resources**: magic on qutrits, coherence with physically incoherent operations
- **Non-convexity**: non-Gaussianity, quantum discord, quantum Markov chain
- **Infinite-dimensionality**: non-Gaussianity

This work: General QRTs with Minimal Axioms

Aim: To investigate manipulation/quantification of as general resources as possible

A QRT is specified by free operations

Axioms on free operations: (no convexity or finite-dimensionality imposed)

1. Free operations can be used at any time in any order. (Closed under **composition**)
2. Free operations can be used regardless of other systems. (Closed under **tensor product**)
3. Doing nothing is free. (The **identity map** is free)
4. Ignoring systems is free. (The **trace** is free)

\+ **Compact sets** of states and free operations

$$\text{Clifford} + T \xrightarrow{\text{closure}} \text{arbitrary } U,$$

$$\text{LOCC} \xrightarrow{\text{closure}} \overline{\text{LOCC}}$$

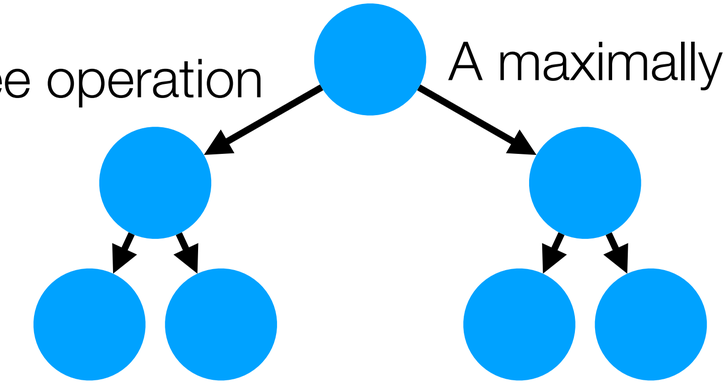
(The choice of free operations specifies free states) $\sigma : \text{ free} \Leftrightarrow \forall \rho \xrightarrow{\text{free operation}} \sigma$

Main results: Universal properties of resources with full **generality** (explained in following slides)

arXiv:2002.02458 arXiv:2103.05665

Result 1/3: Existence of Maximally Resourceful States

Crucial role as the unit in quantifying resources



An order relation

$\mathcal{E}(\rho) = \sigma$, $\mathcal{E}$: free$\Leftrightarrow \rho \succ \sigma$

Maximal = Most resourceful among comparable states

Problem

- **Non-unique** in general: magic on qutrits, coherence with physically incoherent operations
- **Existence is not obvious**: infinite-dimensional bipartite entanglement

Theorem: Maximally resourceful states always exist in all QRTs in our framework.

Proof technique: a mathematical theorem from the theory of topology and ordered sets

+ the assumption of **compactness** of states and free operations

https://arxiv.org/abs/2002.02458

Result 2/3: Catalytic Replication of Resources

A novel form of catalytic transformation in general QRTs

Catalytic replication: asymptotic transformation of a state into many copies

$$\rho \xrightarrow{\text{asymptotic}} \rho^{\otimes r}$$

Using ρ itself as catalyst

r : conversion rate

Proposition: In catalytic replication, $r(\rho \to \rho) = 1 \text{ or } \infty$

Our main contributions: Discovery of **catalytically replicable states**

- Naturally, free states are infinitely replicable
- **Indeed, we discover a case where a resource state is infinitely replicable**

  Example of QRTs with catalytically replicable states: $\{|0\rangle\langle 0|, \mathrm{CNOT}, \mathrm{id}, \mathrm{Tr}\}$, QRT of imaginality

https://arxiv.org/abs/2002.02458

Result 3/3: Consistent Measures of Resources

A good and broadly applicable way of quantifying quantum resources

Resource measure: a family of functions of states with **monotonicity** under free operations

Uniqueness inequality for entanglement [M. J. Donald, M. Horodecki, O. Rudolph (2001)]

Distillable entanglement: Distillation into **ebits**

$$E_{\mathrm{D}}(\rho) \leqq E(\rho) \leqq E_{\mathrm{C}}(\rho)$$

Entanglement cost: Formation from **ebits**

Any measure with **conventional properties**:

normalization, asymptotic continuity, (weak) additivity

Problem: but not simply applicable to QRTs with **non-unique maximally resourceful states**

Result: Uniqueness inequality for general resource measures

Distillable resource: Distillation into the **hardest resource**

$$R_{\mathrm{D}}(\rho) \leqq R(\rho) \leqq R_{\mathrm{C}}(\rho)$$

Resource cost: Formation from the **easiest resource**

(Applicable even to infinite-dimensional cases)

(See also arXiv:2009.11302, arXiv:2009.11313 for another operational robustness measure of infinite-dimensional resources)

https://arxiv.org/abs/2002.02458

Asymptotically Consistent Resource Measures

No resource measure may satisfy the conventional properties simultaneously

Problem (that we found in showing our result):

In QRT of magic on qutrits, no measure satisfies the conventional properties simultaneously

normalization, asymptotic continuity, additivity

→ These properties can be inconsistent in general

Solution: **Asymptotically consistent resource measures**

$R(\rho) \geqq R(\sigma) r(\rho \to \sigma)$: consistent with asymptotic conversion rate $r(\rho \to \sigma)$

Theorem: Any asymptotically consistent measure under an appropriate normalization condition satisfies the uniqueness inequality $R_{\mathrm{D}}(\rho) \leqq R(\rho) \leqq R_{\mathrm{C}}(\rho)$

https://arxiv.org/abs/2002.02458

Wide Applicability of Consistent Measures

Applicable even to QRTs without convexity and finite-dimensionality

Consistency with asymptotic conversion rate $R(\rho) \geqq R(\sigma) r(\rho \to \sigma)$

Example: **Regularized relative entropy of resources** $R_{\mathrm{R}}(\rho) = \lim_{n\to\infty} \frac{1}{n} \min_{\sigma:\ \mathrm{free}} \{D(\rho^{\otimes n} || \sigma)\}$

Wide applicability = A theoretical foundation for quantitative studies

- **All convex and finite-dimensional** QRTs
- **Non-convexity**: quantum discord, quantum Markov chain, (Counterexample in general)
- **Infinite-dimensionality**: non-Gaussianity (for a convex version)

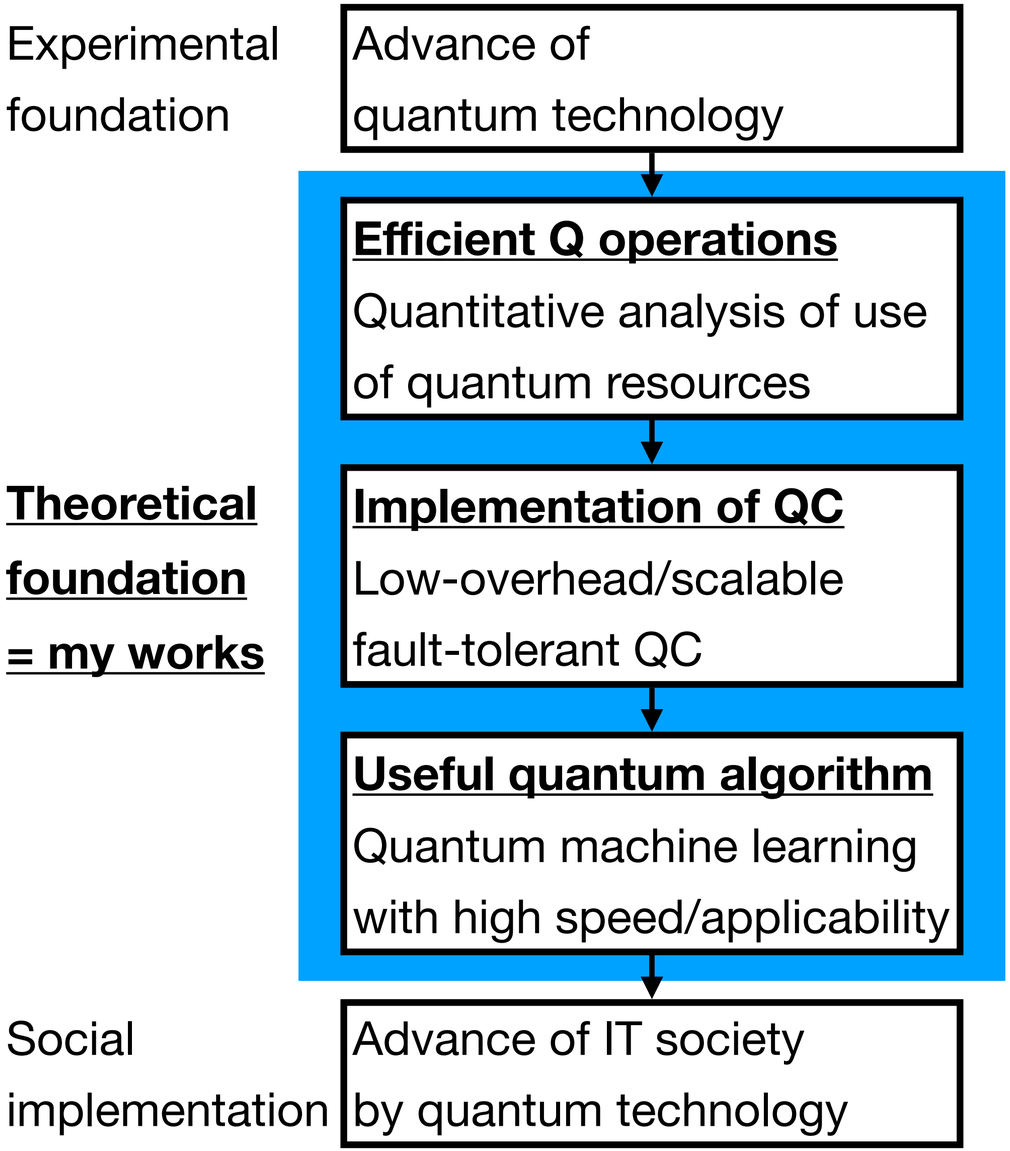
Proof technique: subadditivity & asymptotic continuity of relative entropy of resources

→ A theoretical foundation for quantitative studies of a broad class of quantum properties

https://arxiv.org/abs/2103.05665

Outlook

QRT techniques as a tool for quantitative analysis of quantum info processing



- **General quantum resource theories** arXiv:2002.02458
- **Asymptotically consistent resource measures** arXiv:2103.05665

- QRT analysis of **GKP Code** arXiv:1911.11141
- **Time-efficient constant-space-overhead FTQC** arXiv:2207.08826

- QML using **exponential speedup with avoiding sparse/ low-rank matrices** arXiv:2004.10756
- QML to accelerate **data classification with exponential error convergence** arXiv:2106.09028

https://www.hayatayamasaki.com/

Conclusion

Framework of QRTs in highly general form, covering non-convex & infinite-dimensional cases

Main results

1. Proof of **existence of maximally resourceful states** in general framework
2. Manipulation: **Catalytic replication** of resource states
3. Quantification: Investigation of **asymptotically consistent** resource measures

- **Broadly applicable to QRTs of physically well-motivated resources**

  including magic on qudits, non-Gaussianity, non-Markovianity, and discord
- Foundation for studying a much broader class of quantum properties

  through a unified approach of QRTs

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hayata.yamasaki@gmail.com @hayatayamasaki

Thank you for your attention.